

Few simple rules to fix the dynamics of classical systems using operators

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Abstract

We show how to use operators in the description of *exchanging processes* often taking place in (complex) classical systems. In particular, we propose a set of rules giving rise to an *hamiltonian* operator for such a system \mathcal{S} , which can be used to deduce the dynamics of \mathcal{S}

I Introduction and motivations

In a series of recent papers we have used an operatorial approach in the description of classical systems, with few or with many degrees of freedom, [1]-[8]. In particular, we have shown how canonical commutation and anticommutation relations (CCR and CAR respectively) can be used in the analysis of simplified stock markets, as well as in the description of simpler dynamical systems, like those arising from *love affairs*. We have also adopted the same general settings in the analysis of migration processes and of population dynamics.

The main ingredient in our approach is the *hamiltonian* operator H of the system \mathcal{S} we are interested in, which is used to deduce the time evolution of \mathcal{S} , see below. This paper is devoted to discuss a minimal set of rules which should be adopted to write down H . Some examples of hamiltonians found this way will be discussed. However, the dynamical content of these hamiltonians will not be considered here, since it was already discussed elsewhere, [1]-[8].

The paper is organized as follows: in the rest of this section we review few known facts on CCR. We will not discuss here CAR since they will only play a minor role in Section III.

In Section II we propose our set of rules useful to determine the analytic expression of the hamiltonian of a system \mathcal{S} .

In Section III we discuss few examples, while Section IV contains our conclusions.

The reason why the operator H assumes a crucial role in our approach is because the dynamical behavior of \mathcal{S} is here assumed to be given by the Heisenberg equation of motion:

let \mathcal{H} be an Hilbert space and $B(\mathcal{H})$ the set of all the bounded operators on \mathcal{H} . Let \mathcal{S} be our physical system and \mathfrak{A} the set of all the operators useful for a complete description of \mathcal{S} , which includes the *observables* of \mathcal{S} . The time evolution of \mathcal{S} is assumed to be given by the self-adjoint hamiltonian $H = H^\dagger$ of \mathcal{S} , which in standard quantum mechanics represents the energy of \mathcal{S} . In the Heisenberg picture the time evolution of an observable $X \in \mathfrak{A}$ is expressed by

$$X(t) = e^{iHt} X e^{-iHt} \quad (1.1)$$

or, equivalently, by the solution of the differential equation

$$\frac{dX(t)}{dt} = ie^{iHt} [H, X] e^{-iHt} = i[H, X(t)], \quad (1.2)$$

where $[A, B] := AB - BA$ is the *commutator* between A and B . The time evolution defined in this way is usually a one parameter group of automorphisms of $B(\mathcal{H})$. It might seem that adopting the Heisenberg picture in the description of classical systems may appear inappropriate. However, as discussed in our previous literature as well as in many other papers on

similar subjects, see for instance [9]-[14], this approach is justified *a posteriori* since, at least for simple systems, it produces exactly that time evolution which one expects to find. We should also mention that the uncertainty principle arising from the non abelianity of the operators involved in the description of \mathcal{S} , does not appear in our approach, since all the observables of \mathfrak{A} do commute. Other authors, on the other hand, because of what they are interested in, consider such an uncertainty a richness and not a problem of a quantum view to complex systems, [15].

In our approach a special role is played by the so called CCR: we say that a set of operators $\{a_l, a_l^\dagger, l = 1, 2, \dots, L\}$ satisfy the CCR if the following hold:

$$[a_l, a_n^\dagger] = \delta_{ln} \mathbb{1}, \quad [a_l, a_n] = [a_l^\dagger, a_n^\dagger] = 0, \quad (1.3)$$

for all $l, n = 1, 2, \dots, L$. Here $\mathbb{1}$ is the identity operator on \mathcal{H} . These operators, which are widely analyzed in any textbook in quantum mechanics, see [16] for instance, are those which are used to describe L different *modes* of bosons. From these operators we can construct $\hat{n}_l = a_l^\dagger a_l$ and $\hat{N} = \sum_{l=1}^L \hat{n}_l$ which are both self-adjoint. In particular \hat{n}_l is the *number operator* for the l -th mode, while \hat{N} is the *number operator of \mathcal{S}* .

The Hilbert space of our system is constructed as follows: we introduce the *vacuum* of the theory, that is a vector $\varphi_{0,0,\dots,0}$ which is annihilated by all the operators a_l : $a_l \varphi_{0,0,\dots,0} = 0$ for all $l = 1, 2, \dots, L$. Then we act on $\varphi_{0,0,\dots,0}$ with the operators a_l^\dagger and their powers:

$$\varphi_{n_1, n_2, \dots, n_L} := \frac{1}{\sqrt{n_1! n_2! \dots n_L!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_L^\dagger)^{n_L} \varphi_{0,0,\dots,0}, \quad (1.4)$$

$n_l = 0, 1, 2, \dots$ for all l . These vectors form an orthonormal set and are eigenstates of both \hat{n}_l and \hat{N} : $\hat{n}_l \varphi_{n_1, n_2, \dots, n_L} = n_l \varphi_{n_1, n_2, \dots, n_L}$ and $\hat{N} \varphi_{n_1, n_2, \dots, n_L} = N \varphi_{n_1, n_2, \dots, n_L}$, where $N = \sum_{l=1}^L n_l$. Moreover using the CCR we deduce that

$$\hat{n}_l (a_l \varphi_{n_1, n_2, \dots, n_L}) = (n_l - 1) (a_l \varphi_{n_1, n_2, \dots, n_L}) \quad (1.5)$$

and

$$\hat{n}_l (a_l^\dagger \varphi_{n_1, n_2, \dots, n_L}) = (n_l + 1) (a_l^\dagger \varphi_{n_1, n_2, \dots, n_L}), \quad (1.6)$$

for all l . For these reasons the following interpretation is given: if the L different modes of bosons of \mathcal{S} are described by the vector $\varphi_{n_1, n_2, \dots, n_L}$, this implies that n_1 bosons are in the first mode, n_2 in the second mode, and so on. The operator \hat{n}_l acts on $\varphi_{n_1, n_2, \dots, n_L}$ and returns n_l , which is exactly the number of bosons in the l -th mode. The operator \hat{N} counts the total number of bosons. Moreover, the operator a_l destroys a boson in the l -th mode, while a_l^\dagger creates a boson in the same mode. This is why a_l and a_l^\dagger are usually called the *annihilation* and *creation* operators.

The Hilbert space \mathcal{H} is obtained by taking the closure of the linear span of all these vectors.

A similar construction can be repeated starting with CAR, but we will not consider this possibility here since it is not essential for the general analysis we will discuss in this paper.

An operator $Z \in \mathfrak{A}$ is a *constant of motion* if it commutes with H . Indeed in this case equation (1.2) implies that $\dot{Z}(t) = 0$, so that $Z(t) = Z$ for all t .

The vector $\varphi_{n_1, n_2, \dots, n_L}$ in (1.4) defines a *vector (or number) state* over the algebra \mathfrak{A} as

$$\omega_{n_1, n_2, \dots, n_L}(X) = \langle \varphi_{n_1, n_2, \dots, n_L}, X \varphi_{n_1, n_2, \dots, n_L} \rangle, \quad (1.7)$$

where \langle, \rangle is the scalar product in \mathcal{H} . As we have discussed in [1]-[8], these states are used to *project* from quantum to classical dynamics and to fix the initial conditions of the system.

II The rules

As already discussed in the Introduction, the main interest in this paper is to produce a sort of recipe which has to be used to write down the hamiltonian H of the classical system \mathcal{S} we are interested in. To simplify our analysis, let us first suppose that \mathcal{S} consists of two main interacting parts, \mathcal{S}_1 and \mathcal{S}_2 , the *actors* of the game, whose union reproduces \mathcal{S} and which have no intersection: $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Suppose now that \mathcal{S}_1 and \mathcal{S}_2 can exchange something, \mathfrak{M} , which can only take integer values¹. A typical example of this situation is in stock markets, where two traders \mathcal{S}_1 and \mathcal{S}_2 , exchange money (and shares). Other examples are discussed in [6] and [7], where what is exchanged is *mutual affection* (in other words, love!) between the two lovers. In [8] we have two populations in different regions of a two-dimensional lattice, and they *exchange people*, i.e. there is people moving from one region to the other.

Let us now introduce two annihilation operators, a_1 and a_2 , related respectively to \mathfrak{M}_1 and \mathfrak{M}_2 , and their conjugate creation operators a_1^\dagger and a_2^\dagger . Here \mathfrak{M}_1 is that part of \mathfrak{M} which *belongs* to \mathcal{S}_1 : the money of the first trader, or the number of shares in his portfolio, or jet the amount of love that Bob, the first lover, experiences for Alice, the second one, and so on. As in the Introduction, these operators obey the following CCR: $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$, $[a_i, a_j^\dagger] = \delta_{i,j} \mathbb{I}$. Calling $\varphi_{0,0}$ the vacuum of a_1, a_2 , that is that vector of \mathcal{H} annihilated by a_1 and a_2 , $a_1 \varphi_{0,0} = a_2 \varphi_{0,0} = 0$, the vector $\varphi_{n_1, n_2} := \frac{1}{\sqrt{n_1! n_2!}} a_1^{\dagger n_1} a_2^{\dagger n_2} \varphi_{0,0}$ describes a situation in which the value of \mathfrak{M}_1 is n_1 and that of \mathfrak{M}_2 is n_2 ; indeed, calling $\hat{n}_j := a_j^\dagger a_j$ the related number operators, we know that $\hat{n}_j \varphi_{n_1, n_2} = n_j \varphi_{n_1, n_2}$, $j = 1, 2$. In all the papers written so far, [1]-[8],

¹We could relax this assumption by assuming that the values of \mathfrak{M} are discrete rather than integer.

the eigenvalues of \hat{n}_j , n_j , are considered directly related to the value of \mathfrak{M}_j . Now we are ready to state the first rule of our construction:

Rule 1:—*The exchange of \mathfrak{M} between \mathcal{S}_1 and \mathcal{S}_2 is modeled adding to the hamiltonian of \mathcal{S} a term $a_1^\dagger a_2 + a_2^\dagger a_1$. If, for some reason, the model should be non-linear, then this contribution must be replaced by $a_1^{\dagger M} a_2 + a_2^\dagger a_1^M$, $M > 1$ being a measure of the non linearity.*

The motivation of this rule is given by the action of $a_1^\dagger a_2 + a_2^\dagger a_1$ on the vector φ_{n_1, n_2} :

$$\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) \varphi_{n_1, n_2} \simeq \varphi_{n_1+1, n_2-1} + \varphi_{n_1-1, n_2+1},$$

where the normalization constants are missing since they are not interesting for us, here. As we can see, what we get is a combination of two vectors: the first one, φ_{n_1+1, n_2-1} , shows that the value of \mathfrak{M}_1 is increased by one unit while, simultaneously, \mathfrak{M}_2 decreases by a unit. In the second contribution, φ_{n_1-1, n_2+1} , the opposite happens. In both cases, what it is going on is that \mathcal{S}_1 and \mathcal{S}_2 are exchanging one unit of \mathfrak{M} . Analogously, acting with $a_1^{\dagger M} a_2 + a_2^\dagger a_1^M$ on φ_{n_1, n_2} would produce a combination of vectors φ_{n_1+M, n_2-1} and φ_{n_1-M, n_2+1} , which is useful to introduce a possible asymmetry between \mathcal{S}_1 and \mathcal{S}_2 , or, from a dynamical point of view, a non-linearity in the dynamics of \mathcal{S} , [6].

One may argue why not to add simply $a_1^\dagger a_2$ in the hamiltonian. The reason is the following: if we don't consider both $a_1^\dagger a_2$ and $a_2^\dagger a_1$, the final hamiltonian would not be self-adjoint, and this would create a lot of difficulties in finding a reversible time evolution: for instance, if $H \neq H^\dagger$ then, among other problems, the norm of $e^{iHt} X e^{-iHt}$ is different from that of X , so that the probabilistic interpretation of the wave-function typical of quantum mechanics would be lost. This is obviously related to the decay effects which we don't want to discuss here. We will briefly come back on this aspect in Section IV.

Let us now go the the second rule of our construction:

Rule 2:—*The hamiltonian H for \mathcal{S} must contain a term, H_0 , such that, in absence of interaction between \mathcal{S}_1 and \mathcal{S}_2 , their related number operators, \hat{n}_1 and \hat{n}_2 , stay constant in time.*

This is quite a natural assumption: if \mathcal{S}_1 and \mathcal{S}_2 do not interact, there is no reason for them to modify their situation, and in particular there is no reason (and no possibility!) for exchanging units of \mathfrak{M} . To be concrete, this means that, if at $t = 0$ \mathcal{S} is described by the state φ_{n_1, n_2} , and if no interaction is contained in the hamiltonian, $H = H_0$, then at $t > 0$ the system is still described by φ_{n_1, n_2} (but, at most, for an overall phase). It should be stressed that, however, this does not imply that in this case there is no dynamics at all! What we are

claiming is that $\hat{n}_j(t) = \hat{n}_j(0)$, but this does not imply that, for instance we also necessarily have $a_j(t) = a_j(0)$. On the contrary, in many examples this is not so, see [1] for such an example.

There is still another rule which is quite useful in the determination of H . For that it may be convenient to recall the notion of *closed system*: a system \mathcal{S} is called closed if it has no interaction with the environment \mathcal{R} .

Rule 3:– *If \mathcal{S} is a closed system, the hamiltonian H of \mathcal{S} must commute with those global number-like operators related to the observables which are not exchanged between \mathcal{S} and \mathcal{R} .*

The motivation is, again, rather natural: as we have seen in the Introduction, all the observables which commute with the hamiltonian are *integrals of motion*, so that they do not change with time. This is exactly what is expected to the *global quantities* of the system \mathcal{S} , since they are not moving outside \mathcal{S} . A simple example of this situation is provided by the total number of shares of a certain type in a closed market where the shares are not created or destroyed: if at $t = 0$ this number is $N = n_1 + n_2$, where n_j is the number of shares of that kind which belong to \mathcal{S}_j , then N does not change with time, even if the number of shares in each trader's portfolio does change, in general. In this case, **Rule 3** reads $[H, \hat{n}_1 + \hat{n}_2] = 0$, while $[H, \hat{n}_1] \neq 0$ and $[H, \hat{n}_2] \neq 0$, in general.

III Examples

In this section we will show how the rules described so far can be explicitly used in the analysis of some classical systems, and which kind of hamiltonian are deduced.

III.1 First example: love affair

The first model we have in mind consists of a couple of lovers, Bob and Alice, which mutually *interact* exhibiting a certain *interest* for each other. Of course, there are several degrees of possible interest, and to a given Bob's interest for Alice (LoA, *level of attraction*) there corresponds a related reaction (*i.e.*, a different LoA) of Alice for Bob. In our previous decomposition of \mathcal{S} in \mathcal{S}_1 and \mathcal{S}_2 , here Bob plays the role of \mathcal{S}_1 , while Alice that of \mathcal{S}_2 , and \mathfrak{M} is the mutual affection between the two. The bosonic operators associated to Bob are a_1 , a_1^\dagger and $\hat{n}_1 = a_1^\dagger a_1$, while those associated to Alice are a_2 , a_2^\dagger and $\hat{n}_2 = a_2^\dagger a_2$. The (integer) eigenvalue n_1 of \hat{n}_1 measures the value of the LoA that Bob experiences for Alice: the higher the value of n_1 the more Bob desires Alice. For instance, if $n_1 = 0$, Bob just does not care about Alice. We use

n_2 , the eigenvalue of \hat{n}_2 , to measure the attraction of Alice for Bob. The *law of attraction* we have in mind states that, if n_1 increases, then n_2 decreases and viceversa. This suggests to use the following self-adjoint operator to describe the interaction between Alice and Bob:

$$H = \lambda \left(a_1^M a_2^\dagger + a_2 a_1^{\dagger M} \right), \quad (3.1)$$

where M describes a sort of *relative behavior*, [6]. This choice is written following **Rule 1** of the previous section and it trivially satisfies **Rule 2**: if $\lambda = 0$ there is no dynamics at all since $H = 0$ and, as a consequence, $[H, \hat{n}_1] = [H, \hat{n}_2] = 0$. Concerning **Rule 3**, it is an easy exercise to check that $I(t) := \hat{n}_1(t) + M \hat{n}_2(t)$ is a constant of motion: $I(t) = I(0) = \hat{n}_1(0) + M \hat{n}_2(0)$, for all $t \in \mathbb{R}$, since $[H, I] = 0$. Therefore, during the time evolution, a certain *global attraction* is preserved and it can only be exchanged between Alice and Bob: notice that this reproduces our original point of view on the love relation between Alice and Bob: the more Bob falls in love with Alice, the less Alice cares about Bob! If M is fixed to be one then Bob and Alice react in the same way and the model becomes exactly solvable, [6].

In [6] we have also considered a love affair involving, other than Alice and Bob, a third actress, Carla, also having a relation with Bob. Our assumptions are the following: (1) Bob can interact with both Alice and Carla, but Alice (respectively, Carla) does not suspect of Carla's (respectively, Alice's) role in Bob's life; (2) if Bob's LoA for Alice increases then Alice's LoA for Bob decreases and viceversa; (3) analogously, if Bob's LoA for Carla increases then Carla's LoA for Bob decreases and viceversa; (4) if Bob's LoA for Alice increases then his LoA for Carla decreases (not necessarily by the same amount) and viceversa.

Introducing now the operators a_3 , a_3^\dagger and $\hat{n}_3 = a_3^\dagger a_3$ for Carla, and splitting the operators related to Bob in two (i.e. a_{12} and a_{13} to describe the interaction between Bob and, respectively, Alice and Carla) the hamiltonian which describes all these effects is the following:

$$H = \lambda_{12} \left(a_{12}^\dagger a_2 + a_{12} a_2^\dagger \right) + \lambda_{13} \left(a_{13}^\dagger a_3 + a_{13} a_3^\dagger \right) + \lambda_1 \left(a_{12}^\dagger a_{13} + a_{12} a_{13}^\dagger \right), \quad (3.2)$$

for some real values of λ 's. It can be easily seen that the first contribution, $\lambda_{12} \left(a_{12}^\dagger a_2 + a_{12} a_2^\dagger \right)$, describes the mechanism (2) above, while $\lambda_{13} \left(a_{13}^\dagger a_3 + a_{13} a_3^\dagger \right)$ is related to point (3). Point (4) is implemented by $\lambda_1 \left(a_{12}^\dagger a_{13} + a_{12} a_{13}^\dagger \right)$. These three contributions all trivially satisfy **Rule 1** and, again, **Rule 2** is also verified: no interaction means that all the λ 's are zero, so that H reduces to the zero operator, and all the observables stay constant in time. Let us now introduce $\hat{n}_{12} = a_{12}^\dagger a_{12}$, describing Bob's LoA for Alice, $\hat{n}_{13} = a_{13}^\dagger a_{13}$, describing Bob's LoA for Carla, $\hat{n}_2 = a_2^\dagger a_2$, describing Alice's LoA for Bob and $\hat{n}_3 = a_3^\dagger a_3$, describing Carla's LoA

for Bob. If we define $J := \hat{n}_{12} + \hat{n}_{13} + \hat{n}_2 + \hat{n}_3$, which represents the global level of LoA of the triangle, this is a conserved quantity: $J(t) = J(0)$, since $[H, J] = 0$: no exchange with the environment is possible, here! It is also possible to check that $[H, \hat{n}_{12} + \hat{n}_{13}] \neq 0$, so that the total Bob's LoA is not conserved during the time evolution.

More details, as the equations of motion arising from these hamiltonians, their solutions and more extensions can be found in [6] and [7].

III.2 Second example: competition between species and migration

In this example we consider a two-dimensional region \mathcal{R} in which two populations \mathcal{S}_1 and \mathcal{S}_2 are distributed. In [8] we have considered these species as predators and preys, or as two migrant populations, moving from one part of \mathcal{R} to another. Following the above rules we can construct the hamiltonian of the full system \mathcal{S} . However, for reasons discussed in [8], it is convenient to use here annihilation and creation operators satisfying CAR rather than CCR. This choice is motivated by a first technical and a second more substantial reason: the technical reason is that we get finite dimensional Hilbert space for \mathcal{S} , while the more substantial reason is that, using CAR, we automatically incorporate an upper bound for the densities of the two populations, which is a natural requirement for our *biological* interpretation.

The starting point is the (*e.g.*, rectangular or square) region \mathcal{R} , which we divide in N cells, labeled by $\alpha = 1, 2, \dots, N$. In each cell α the two populations, whose related operators are a_α , a_α^\dagger and $\hat{n}_\alpha^{(a)} = a_\alpha^\dagger a_\alpha$ for what concerns \mathcal{S}_1 , and b_α , b_α^\dagger and $\hat{n}_\alpha^{(b)} = b_\alpha^\dagger b_\alpha$ for \mathcal{S}_2 , are described by

$$H_\alpha = H_\alpha^0 + \lambda_\alpha H_\alpha^I, \quad H_\alpha^0 = \omega_\alpha^a a_\alpha^\dagger a_\alpha + \omega_\alpha^b b_\alpha^\dagger b_\alpha, \quad H_\alpha^I = a_\alpha^\dagger b_\alpha + b_\alpha^\dagger a_\alpha. \quad (3.3)$$

It is natural to interpret the mean values of the operators $\hat{n}_\alpha^{(a)}$ and $\hat{n}_\alpha^{(b)}$ as *local density operators* of the two populations in the cell α : if the mean value of, say, $\hat{n}_\alpha^{(a)}$, in the state of the system is equal to one, this means that the density of \mathcal{S}_1 in the cell α is very high. Notice that $H_\alpha = H_\alpha^\dagger$, since all the parameters, which in general are assumed to be cell-dependent (to allow for the description of an anisotropic situation), are real and positive numbers. The CAR are

$$\{a_\alpha, a_\beta^\dagger\} = \{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha,\beta} \mathbb{1}, \quad \{a_\alpha^\dagger, b_\beta^\dagger\} = 0. \quad (3.4)$$

Of course, the full hamiltonian H must consist of a sum of all the different H_α plus another contribution, H_{diff} , responsible for the diffusion of the populations all around the lattice. A natural choice for H_{diff} , in view of the above rules, is the following:

$$H_{diff} = \sum_{\alpha,\beta} p_{\alpha,\beta} \left\{ \gamma_a \left(a_\alpha a_\beta^\dagger + a_\beta a_\alpha^\dagger \right) + \gamma_b \left(b_\alpha b_\beta^\dagger + b_\beta b_\alpha^\dagger \right) \right\}, \quad (3.5)$$

where also γ_a, γ_b and the $p_{\alpha,\beta}$ are real quantities. In particular, $p_{\alpha,\beta}$ can only be 0 or 1 depending on the possibility of the populations to move from cell α to cell β or vice-versa. For this reason they are considered as *diffusion coefficients*. Notice that a similar role is also played by γ_a and γ_b . $H = \sum_{\alpha} H_{\alpha} + H_{diff}$ obeys the three rules of Section II. Indeed, if there is no interaction between \mathcal{S}_1 and \mathcal{S}_2 , and between members of the same species localized in different cells of \mathcal{R} , it is easy to check that the densities of \mathcal{S}_1 and \mathcal{S}_2 stay constant in all the cells: $\hat{n}_{\alpha}^{(a)}(t) = \hat{n}_{\alpha}^{(a)}(0)$ and $\hat{n}_{\alpha}^{(b)}(t) = \hat{n}_{\alpha}^{(b)}(0)$, for all α . Hence **Rule 2** holds true. **Rule 3** is also satisfied, since \mathcal{S}_1 and \mathcal{S}_2 cannot move outside \mathcal{R} : it is again possible to find an operator, related to the total number of members of \mathcal{S} distributed all along \mathcal{R} , which commutes with H , so that this *global density* stays constant in time. Concerning **Rule 1**, we see that this is applied several times in the definition of H . For instance we have the contribution $a_{\alpha}^{\dagger}b_{\alpha} + b_{\alpha}^{\dagger}a_{\alpha}$, which shows how **Rule 1** is applied in the interaction between \mathcal{S}_1 and \mathcal{S}_2 in the cell α , but we also have $a_{\alpha}a_{\beta}^{\dagger} + a_{\beta}a_{\alpha}^{\dagger}$, which is again **Rule 1**, but applied to \mathcal{S}_1 in different cells. And so on. Again, we refer to [8] for the analysis of the equations of motion arising from this hamiltonian.

III.3 Last example: stock market

In recent years we have proposed several hamiltonians describing simplified stock markets, [1]-[4]. The one we discuss here, the *most efficient* proposal, so far, was first introduced in [4].

Let us consider N different traders $\tau_1, \tau_2, \dots, \tau_N$, exchanging L different kind of shares $\sigma_1, \sigma_2, \dots, \sigma_L$. Each trader has a starting amount of cash, which is used during the trading procedure: the cash of the trader who sells a share increases while the cash of the trader who buys that share consequently decreases. The absolute value of these variations is the price of the share at the time in which the transaction takes place. It is clear that the above-mentioned division of \mathcal{S} in just two plus one components, $\mathcal{S}_1, \mathcal{S}_2$ and \mathfrak{M} , must be extended here, while the main ideas are unchanged. It is convenient to introduce a set of bosonic operators which are listed, together with their economical meaning, in the following table. We are adopting here latin indexes to label the traders and greek indexes for the shares: $j = 1, 2, \dots, N$ and $\alpha = 1, 2, \dots, L$.

	the operator and..	...its economical meaning
	$a_{j,\alpha}$	annihilates a share σ_α in the portfolio of τ_j
	$a_{j,\alpha}^\dagger$	creates a share σ_α in the portfolio of τ_j
	$\hat{n}_{j,\alpha} = a_{j,\alpha}^\dagger a_{j,\alpha}$	counts the number of share σ_α in the portfolio of τ_j
	c_j	annihilates a monetary unit in the portfolio of τ_j
	c_j^\dagger	creates a monetary unit in the portfolio of τ_j
	$\hat{k}_j = c_j^\dagger c_j$	counts the number of monetary units in the portfolio of τ_j
	p_α	lowers the price of the share σ_α of one unit of cash
	p_α^\dagger	increases the price of the share σ_α of one unit of cash
	$\hat{P}_\alpha = p_\alpha^\dagger p_\alpha$	gives the value of the share σ_α

Table 1.– List of operators and of their *economical* meaning.

These operators are bosonic in the sense that they satisfy the following commutation rules

$$[c_j, c_k^\dagger] = \mathbb{1} \delta_{j,k}, \quad [p_\alpha, p_\beta^\dagger] = \mathbb{1} \delta_{\alpha,\beta} \quad [a_{j,\alpha}, a_{k,\beta}^\dagger] = \mathbb{1} \delta_{j,k} \delta_{\alpha,\beta}, \quad (3.6)$$

while all the other commutators are zero. We assume that the hamiltonian of the market, \hat{H} , can be written as $\hat{H} = H + H_{prices}$, where

$$\begin{cases} H = H_0 + \lambda H_I, \text{ with} \\ H_0 = \sum_{j,\alpha} \omega_{j,\alpha} \hat{n}_{j,\alpha} + \sum_j \omega_j \hat{k}_j \\ H_I = \sum_{i,j,\alpha} p_{i,j}^{(\alpha)} \left(a_{i,\alpha}^\dagger a_{j,\alpha} c_i^{\dagger \hat{P}_\alpha} c_j^{\dagger \hat{P}_\alpha} + h.c. \right). \end{cases} \quad (3.7)$$

Here h.c. stands for hermitian conjugate, $c_i^{\hat{P}_\alpha}$ and $c_j^{\dagger \hat{P}_\alpha}$ are defined as in [2], and $\omega_{j,\alpha}$, ω_j and $p_{i,j}^{(\alpha)}$ are positive real numbers. In particular these last coefficients assume different values depending on the possibility of τ_i to interact with τ_j and exchanging a share σ_α : for instance $p_{2,5}^{(1)} = 0$ if there is no way for τ_2 and τ_5 to exchange a share σ_1 . It is natural to put $p_{i,i}^{(\alpha)} = 0$ and $p_{i,j}^{(\alpha)} = p_{j,i}^{(\alpha)}$.

Going back to (3.7), we observe that H obeys **Rule 2** of Section 2, since, if there is no interaction between the traders, then $\lambda = 0$ and, as a consequence, $H = H_0$: $[H_0, \hat{n}_{j,\alpha}] = 0$, for all j and α . As for H_I , this is written obeying **Rule 1**: the action of a single contribution of H_I , $a_{i,\alpha}^\dagger a_{j,\alpha} c_i^{\dagger \hat{P}_\alpha} c_j^{\dagger \hat{P}_\alpha}$, on a vector number which extends those introduced in Section II, $\varphi_{\{n_{j,\alpha}\};\{k_j\};\{P_\alpha\}}$, is proportional to another vector $\varphi_{\{n'_{j,\alpha}\};\{k'_j\};\{P'_\alpha\}}$ with just 4 different quantum numbers. In particular $n_{j,\alpha}$, $n_{i,\alpha}$, k_j and k_i are replaced respectively by $n_{j,\alpha} - 1$, $n_{i,\alpha} + 1$, $k_j + P_\alpha$ and $k_i - P_\alpha$ (if this is larger or equal than zero, otherwise the vector is annihilated). This means that τ_j is

selling a share σ_α to τ_i and earning money from this operation. For this reason it is convenient to introduce the following *selling* and *buying* operators:

$$x_{j,\alpha} := a_{j,\alpha} c_j^{\dagger \hat{P}_\alpha}, \quad x_{j,\alpha}^\dagger := a_{j,\alpha}^\dagger c_j^{\hat{P}_\alpha} \quad (3.8)$$

With these definitions and using the properties of the coefficients $p_{i,j}^{(\alpha)}$ we can rewrite H_I as

$$H_I = 2 \sum_{i,j,\alpha} p_{i,j}^{(\alpha)} x_{i,\alpha}^\dagger x_{j,\alpha} \Rightarrow H = \sum_{j,\alpha} \omega_{j,\alpha} \hat{n}_{j,\alpha} + \sum_j \omega_j \hat{k}_j + 2\lambda \sum_{i,j,\alpha} p_{i,j}^{(\alpha)} x_{i,\alpha}^\dagger x_{j,\alpha} \quad (3.9)$$

The role of H_{price} in [4] was to fix the time evolution of the operators \hat{P}_α , $\alpha = 1, 2, \dots, L$. The hamiltonian \hat{H} corresponds to a closed market where the money and the total number of shares of each type are conserved. Indeed, calling $\hat{N}_\alpha := \sum_{l=1}^N \hat{n}_{l,\alpha}$ and $\hat{K} := \sum_{l=1}^N \hat{k}_l$ we see that, for all α , $[\hat{H}, \hat{N}_\alpha] = [\hat{H}, \hat{K}] = 0$. Hence \hat{N}_α and \hat{K} are integrals of motion, as expected: **Rule 3** is satisfied. We refer to [4] for the analysis of the time evolution of the *portfolio operator* of the trader τ_l , $\hat{\Pi}_l(t) = \sum_{\alpha=1}^L \hat{P}_\alpha(t) \hat{n}_{l,\alpha}(t) + \hat{k}_l(t)$.

IV Further considerations and conclusions

The same rules have already been adopted for systems which are not closed, i.e. for those systems which exchange *something* with the environment. This is discussed, for instance, in [7]: again, the main idea is that we can use creation and annihilation operators also in the description of the reservoir, and in modeling an exchange between the system and the reservoir. This exchange is described adding in the hamiltonian a contribution obeying **Rule 1**, while **Rule 2** has to be intended here in the following way: if \mathcal{S} does not interact with the reservoir, no decay is allowed. **Rule 3** is recovered for some global quantity which mixes the degrees of freedom of the reservoir and of the system. The conclusion is, therefore, that our rules can be used also in more general, and sometimes more useful, contexts.

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